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Convection-driven quadrupolar dynamos in rotating spherical shells

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It is found that for Taylor numbers of the order 10^8 quadrupolar dynamos aligned with the axis of rotation are preferred in comparison with dipolar dynamos. This preference holds for a range of Prandtl numbers P and magnetic Prandtl numbers P_m in the neighborhood of unity. The main time-dependent feature of the quadrupolar dynamos are polward traveling waves.
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I. INTRODUCTION

The rapid increase of computer capacities in recent years has permitted a considerable increase in the parameter regime for which the problem of magnetic field generation in rotating spherical shells can be solved by numerical simulations. While many recent efforts have been directed towards simulations of the geodynamo operating in the liquid iron core of the Earth [1–3], other numerical simulations have been directed towards simpler configurations in order to promote a deeper understanding of convection-driven dynamos as a function of the most important physical parameters [4–8]. In the course of this research usually a preference for dipolar dynamos has been found. In this Rapid Communication we demonstrate the surprising phenomenon that quadrupolar dynamos become preferred as the Taylor number is increased.

II. MATHEMATICAL DESCRIPTION OF THE DYNAMO PROBLEM

We use a standard formulation of the spherical dynamo problem with a minimum number of physical parameters that include those of primary importance for planetary applications. While it is not possible to reach in simulations the asymptotically high values of the dimensionless parameters realized in the planets, it seems feasible to attain appropriate values of parameters such as the Rayleigh number and the Taylor number if the latter are based on eddy diffusivities instead of molecular diffusivities.

In considering a spherical fluid shell of thickness d which is rotating about a fixed axis with the constant angular velocity Ω we assume that a static state exists with the temperature distribution $T_S = T_0 - \beta d^2 r^2/2$. As is appropriate for a sphere of approximately uniform density, the gravity field is given by $\vec{g} = -\gamma d\vec{r}$ where $\vec{r}$ is the position vector with respect to the center of the sphere and r is its length measured in units of d. In addition to the length d the time d^2/ν and the temperature $\beta d^2\nu/\kappa$ will be used as scales for a dimensionless description of the problem where ν denotes the kinematic viscosity of the fluid and κ its thermal diffusivity. The density $\vec{B}$ of magnetic flux is made dimensionless by $\nu(\mu\varrho)^{1/2}/d$ where μ is the magnetic permeability of the fluid and ϱ is its density. The latter will be assumed as a constant except in the gravity term where its temperature dependence given by $\alpha \equiv (d\varrho/dT)/\varrho = \text{const}$ is taken into account. Since the velocity field $\vec{u}$ as well as the magnetic field $\vec{B}$ are solenoidal vector fields, we use the general representation in terms of poloidal and toroidal components,

$$\vec{u} = \vec{\nabla} \times (\vec{\nabla} v \times \vec{r}) + \vec{\nabla} w \times \vec{r}, \tag{1a}$$

$$\vec{B} = \vec{\nabla} \times (\vec{\nabla} h \times \vec{r}) + \vec{\nabla} g \times \vec{r}. \tag{1b}$$

By multiplying the (curl)^2 and the curl of the Navier-Stokes equations in the rotating system by $\vec{r}$ we obtain two equations for v and w,

$$\begin{aligned}[(\nabla^2 - \partial_t)L_2 + \tau\partial_\varphi]\nabla^2 v + \tau Q w - R L_2 \Theta \\ = -\vec{r}\cdot\vec{\nabla}\times[\vec{\nabla}\times(\vec{u}\cdot\vec{\nabla}\vec{u} - \vec{B}\cdot\vec{\nabla}\vec{B})],\end{aligned} \tag{2a}$$

$$[(\nabla^2 - \partial_t)L_2 + \tau\partial_\varphi]w - \tau Q v = \vec{r}\cdot\vec{\nabla}\times(\vec{u}\cdot\vec{\nabla}\vec{u} - \vec{B}\cdot\vec{\nabla}\vec{B}), \tag{2b}$$

where ∂_t and ∂_φ denote the partial derivatives with respect to time t and azimuthal angle φ. The heat equation for the dimensionless deviation Θ from the static temperature distribution can be written in the form

$$\nabla^2\Theta + L_2 v = P(\partial_t + \vec{u}\cdot\vec{\nabla})\Theta \tag{2c}$$

and the equations for h and g are obtained through the multiplication of the equation of magnetic induction and of its curl by $\vec{r}$,

$$\nabla^2 L_2 h = P_m[\partial_t L_2 h - \vec{r}\cdot\vec{\nabla}\times(\vec{u}\times\vec{B})], \tag{2d}$$

$$\nabla^2 L_2 g = P_m\{\partial_t L_2 g - \vec{r}\cdot\vec{\nabla}\times[\vec{\nabla}\times(\vec{u}\times\vec{B})]\}. \tag{2e}$$

The operators L_2 and Q are defined by

$$L_2 \equiv -r^2\nabla^2 + \partial_r(r^2\partial_r), \tag{3a}$$

$$Q \equiv r\cos\theta\nabla^2 - (L_2 + r\partial_r)(\cos\theta\partial_r - r^{-1}\sin\theta\partial_\theta), \tag{3b}$$

where a spherical system of coordinates r, θ, φ has been introduced. The Rayleigh number R, the Taylor number τ^2, the Prandtl number P, and the magnetic Prandtl number P_m are defined by

$$R=\frac{\alpha\gamma\beta d^6}{\nu\kappa}, \quad \tau=\frac{2\Omega d^2}{\nu}, \quad P=\frac{\nu}{\kappa}, \quad P_m=\frac{\nu}{\lambda}, \tag{4}$$

where λ is the magnetic diffusivity. We assume stress-free boundaries with fixed temperatures,

$$v=\partial_{rr}^2 v=\partial_r(w/r)=\Theta=0 \text{ at } \quad r=r_i\equiv\eta/(1-\eta) \quad \text{and at } r=r_0=(1-\eta)^{-1}, \tag{5a}$$

where η is the radius ratio, $\eta=r_i/r_o$, which will be set to 0.4 unless indicated otherwise. For the magnetic field electrically insulating boundaries are used such that a matching to potential fields $h^{(e)}$ outside the fluid shell is required

$$g=h-h^{(e)}=\partial_r(h-h^{(e)})=0 \text{ at } r=r_i \text{ and } r=r_0. \tag{5b}$$

The numerical solution of the problem defined by equations (2) and boundary conditions (5) proceeds with the pseudo-spectral method described in [7], which is based on an expansion of all dependent variables in spherical harmonics for the θ,φ-dependences and expansions in Chebychev polynomials for the r-dependence. For further details, see also [9].

For the computations to be reported in the following 33 collocation points in the radial direction and spherical harmonics up to the order 64 have been used. Usually only even values m of the azimuthal wave number have been included such that velocity and magnetic fields exhibit a period of 180° in longitude.

III. QUADRUPOLAR DYNAMOS

Solutions of the problem formulated in the preceding section exhibiting quadrupolar dynamos have been obtained for $P=1$, $P_m=1$ in the regime $3\times10^3\leqslant\tau\leqslant1.2\times10^4$. The corresponding Rayleigh numbers increase from about 3×10^5 to 10^6 within this regime. The energy of the dipolar component of the magnetic field which in typical runs was initially of the same size or larger than the energy of the quadrupolar component decreased to less than 10^{-6} of the latter within several magnetic diffusion times. Quadrupolar dynamos have also been found when $P_m=2$ was used or when η was changed to 0.35 or to 0.5 or when all integer values m of the azimuthal wave number were included. In order to characterize the dynamos we follow the kinetic energy densities $\overline{E}_p,\overline{E}_t,\check{E}_p,\check{E}_t$ which are defined by

$$\overline{E}_p=\frac{1}{2}\langle|\vec{\nabla}\times(\vec{\nabla}\overline{v}\times\vec{r})|^2\rangle, \quad \overline{E}_t=\frac{1}{2}\langle|\vec{\nabla}\overline{w}\times\vec{r}|^2\rangle, \tag{6a}$$

$$\check{E}_p=\frac{1}{2}\langle|\vec{\nabla}\times(\vec{\nabla}\check{v}\times\vec{r})|^2\rangle, \quad \check{E}_t=\frac{1}{2}\langle|\vec{\nabla}\check{w}\times\vec{r}|^2\rangle, \tag{6b}$$

where the brackets $\langle\ldots\rangle$ indicate the average over the spherical shell, $\overline{v}$ denotes the axisymmetric component of v and $\check{v}\equiv v-\overline{v}$ denotes the azimuthally fluctuating component of v. The corresponding magnetic energies $\overline{M}_p,\overline{M}_t,\check{M}_p,\check{M}_t$ are defined analogously with h and g replacing v and w. All solutions found so far are turbulent and a typical example of the variations in time of the energies is shown in Fig. 1. Usually integrations until a time of the order 10 are sufficient to reach a statistically asymptotic state. The rather rapid oscillations seen in the energies are the result of the oscillatory nature of the quadrupolar dynamos. The dynamo process functions as a wave in which patches of magnetic field with the same polarity move from the equator towards the poles as shown in Fig. 2. This oscillation is also clearly evident in the structure of the azimuthal magnetic field $\overline{B}_\varphi$. As before, the bar denotes the axisymmetric component of B_φ. Because of the nearly perfect symmetry with respect to the equatorial plane only the structure in the northern hemisphere has been plotted. The convection velocity field retains its columnar

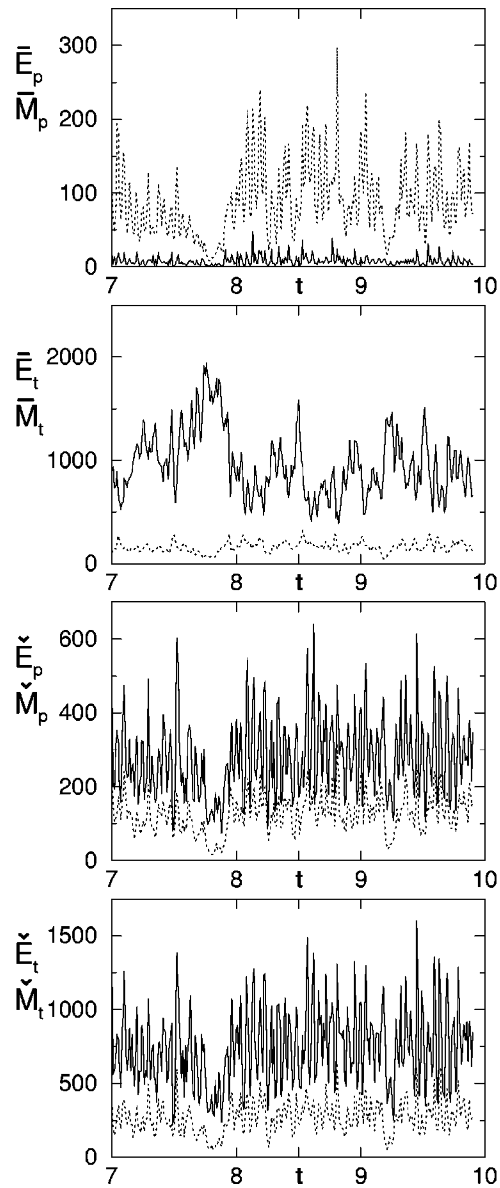

FIG. 1. Energy densities $\overline{E}_p,\overline{E}_t,\check{E}_p,\check{E}_t$ (from top to bottom) of the velocity field (solid lines) and corresponding magnetic energy densities $\overline{M}_p,\overline{M}_t,\check{M}_p,\check{M}_t$ (dashed lines) as a function of time t in the case $\tau=10^4, R=8\times10^5, P_m=P=1$.

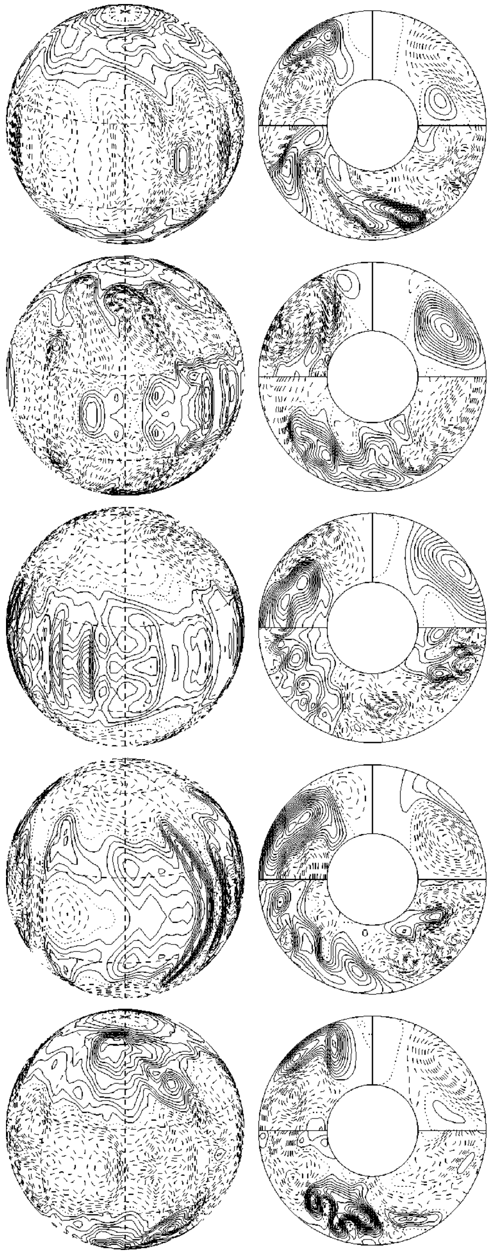

FIG. 2. Time sequence of plots covering approximately a period of the quadrupole oscillations at the times $t=9.61+0.025n, n=0,1,2,3,4$ (from top to bottom) in the same case as in Fig. 1. The left plots show lines of constant B_r at the surface $r=r_0$. The right plots show lines of constant $\overline{B_\varphi}$ (upper left quarter), field lines of $\overline{\vec{B}}$ in the meriodional plane (upper right quarter), and field lines given by constant $r\partial h/\partial\varphi$ in the equatorial plane.

structure as shown in Fig. 3. This property is a result of the dominating influence of the Coriolis force and has often been demonstrated in cases without magnetic field [9,10] or in the presence of a predominantly dipolar magnetic field [7,8]. The mechanism of magnetic field generation appears to be primarily of the α^2-type as in the case of dipolar dynamos [6,8]. This is unusual since dynamos exhibiting propagating waves as shown in Fig. 2 are usually associated with the $\alpha\omega$-process in which the mean zonal magnetic field is generated by the action of a differential rotation. For a recent paper on this topic and earlier references, see [11]. In the

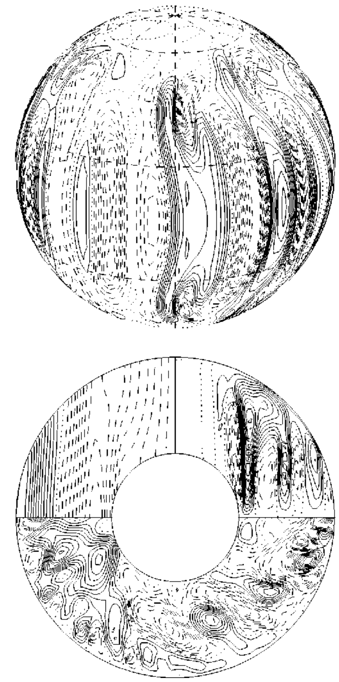

FIG. 3. Lines of constant radial component u_r of the velocity field $\vec{u}$ at the surface $r=(1+\eta)/2(1-\eta)$ (upper plot) and lines of constant $\overline{u_\varphi}$ (upper left part of the lower plot), of constant $r\sin\theta\partial\overline{v}/\partial\theta$ (upper right part of the lower plot), and of constant $r\partial v/\partial\varphi$ in the equatorial plane (lower half of lower plot) for the case of Fig. 2 at the time $t=9.66$.

present case it turns out that the term involving $\check{w}$ and $\check{g}$ on the right-hand side of Eq. (2e) is the main contributor to the sustenance of $\overline{M}_t$. The zonal flow, on the other hand, tends to inhibit the growth of the magnetic field, as can be seen in Fig. 1 where the kinetic energy $\overline{E}_t$ is negatively correlated with the magnetic energies.

The preference of quadrupolar dynamos appears to be a result of the suppression of convection in the polar regions at high values of τ. At lower τ values convection in the polar region is also suppressed, but only for a smaller range of supercritical values of R for which the amplitude of convection is not sufficient to generate magnetic fields. We thus find dynamos of mixed type with dipolar fields primarily in the polar regions and quadrupolar fields at lower latitudes at Rayleigh numbers of the order 5×10^5 when τ is decreased to 3×10^3. At even lower values of τ the dynamos are predominantly dipolar. Some more details on quadrupolar dynamos will be given in a future paper [12].

IV. CONCLUDING REMARKS

The close competition between dynamos of dipolar and quadrupolar symmetry in the case of velocity fields that are symmetric with respect to the equatorial plane is a well known property of kinematic dynamo theory [13]. The question whether quadrupolar dynamos are actually realized in celestial bodies is an open one and it may be difficult to answer. Just as the geodynamo is characterized by a substantial quadrupolar component in addition to its dominant dipolar magnetic field, a quadrupolar stellar or planetary dynamo must be expected to have also a considerable dipolar component. Since at a sufficiently large distance from the planetary core or the star, the dipolar component always dominates, a quadrupolar dynamo could be mistaken for a dipolar one. The possibility that axial quadrupolar dynamos operate in the interiors of Uranus and Neptune could explain the awkward dipole axis inferred from the Voyager measurements [14,15], which has been a rather puzzling feature ever since its discovery. But this possibility appears to be remote according to a recent analysis of the data [16].

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